Controller Tuning by a Least-Squares Method

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Various controller tuning methods are described in the literature. Earlier methods concentrated on a particular aspect of the closed-loop dynamic response, such as quarter decay ratio, maximum overshoot, settling time, and others. With the advent of computers, it became possible to determine controller settings that minimized some integral criterion: Integral Absolute Error (IAE), Integral Square Error (ISE), Integral Time Absolute Error (ITAE), etc. Gallier and Otto (1968) used the direct search procedure of Fletcher and Powell (1963) to find PI and PID controller constants that minimized the IAE criterion for a second-order plus dead-time process model. Fertik (1975) used a similar method to obtain controller settings that minimize the ITAE criterion. Both these researchers developed graphs that give the controller constants as functions of the process parameters. More recently, Ralston et al. (1985) applied the adaptive random search optimization algorithm (Luus and Jaakola, 1973) to the process identification and controller tuning problem. This procedure has the ability to converge to the optimum set of tuning parameters from a wide range of starting values.

In this note we describe a method to find the optimum PID controller tuning parameters based on the least-squares estimation method of Gauss and Legendre (Beightler et al., 1979) as modified by Marquardt (1963). This algorithm exploits the specific structure of the objective function to be minimized (sum of the squares of the errors). As a result, this algorithm converges to the optimum much faster than Fletcher and Powell's method (Beightler et al., 1979). We also compare execution times of our algorithm with those of Ralston et al. (1985). It is assumed that the process parameters have been determined earlier by a suitable identification procedure.

Problem Statement

Consider a typical feedback control loop. When a step change is introduced in the set point, one would ideally desire the process output to exhibit a step change. Due to process dynamics, however, the actual process output increases gradually, oscillates, and then settles down to the new set point. Our goal is to find that set of feedback controller tuning constants which drives the process output as close to a step change as possible, while minimizing one of the integral criteria (IAE, ISE, ITAE, etc.).

The Least-Squares Algorithm

The least-squares curve fitting method of Guass and Legendre seeks to minimize objective functions of the kind

$$y(x) = \sum_{m=1}^{M} g_m^2 \tag{1}$$

Each g_m represents the difference between an experimentally measured value and a prediction based on a set of adjustable parameters, x. Let G denote the column vector of M functions $g_m(x)$ so that

$$y = G^T G \tag{2}$$

In order to minimize y, we require that the gradient of y must equal zero. Using a Taylor series approximation, Guass showed that ∇y becomes zero at a point $x + \Delta x$, where Δx is given by

$$\Delta x = -(J^T J)^{-1} J^T G \tag{3}$$

Here, J is the $M \times N$ Jacobian matrix. This method is potentially divergent, however, and Marquardt recommends using a Δx that is an interpolation between that for a Taylor series method and the gradient method. He modifies Eq. 3 to

$$\Delta x = -(J^T J + \lambda I)^{-1} J^T G \tag{4}$$

where λ is a positive real number. Marquardt has described a method to find a λ that guarantees convergence. The Δx calculated using Eq. 3 gives $x + \Delta x$, which is the set of adjustable parameters for the next iteration. When J^TG becomes zero, we halt iterations because ∇y is zero.

Application to Controller Tuning

The quantity g_m is a suitably defined function of the error, e_m, depending upon the integral criterion to be minimized. For example, if one wants to minimize the ISE criterion,

$$g_m = e_m \tag{5}$$

while

$$g_m = \text{sgn}(e_m)\sqrt{e_m} \tag{6}$$

to minimize the IAE criterion, and

$$g_m = \text{sgn}(e_m)\sqrt{me_m} \tag{7}$$

to minimize the ITAE criterion. The controller tuning constants K_C, τ_I, and τ_D constitute the set of adjustable parameters, x. The elements of the Jacobian, J, are approximated by a finite difference

$$\frac{\partial g_m}{\partial x_l} = \frac{g_m(x_1 + \Delta x_1, x_2, x_3) - g_m(x_1, x_2, x_3)}{\Delta x_l} \tag{8}$$

The partial derivatives of g_m with respect to x_2 and x_3 are evaluated in a similar manner.

Results and Discussion

The algorithm described above was implemented on a CYBER 835 computer. The optimum controller parameters calculated by the least-squares algorithm are presented in Table 1. It is seen that the algorithm can converge from a wide variety of starting values for the three tuning constants. For the initial guesses for cases (b) and (c) in the table, the number of iterations for convergence is high because they are in the unstable region. However, it is generally possible to select stable tuning constants (low K_C, high τ_I, and low τ_D) as in case (d), and reduce the number of iterations required. Results from the Luus-Jaakola (L-J) method are also presented for comparison. In all instances, the least-squares algorithm converged to the minimum IAE in much smaller computation times than the L-J algorithm. In the L-J algorithm, we employed an initial range of 10 for each parameter, a range reduction factor of 0.2, and 100 cycles per iteration (Ralston et al., 1985). For the least-squares algorithm, the initial value of λ was taken to be 0.01 and the multiplier was 5 (Marquardt, 1963).

Figure 1 shows the progress toward convergence of the least-squares and the L-J algorithms for one set of starting values. It is seen that the L-J algorithm converges very rapidly in the first couple of iterations, but then slows down drastically. It may therefore be advantageous to employ a combination of the two procedures, where the first one or two iterations are performed using the L-J algorithm, and then a switch is made to the least-squares method. We have done this and found the results to be superior in most cases. In the combined algorithm, the first iteration was performed using the L-J algorithm and then a switch made to the least-squares method. This not only reduces the CPU time, as seen from Table 1, but also reduces the number of iterations (in most cases), as is evident from Figure 1. Only in case (d) did the combined method take more computational time than the least-squares method. This is due to the fact that the first iteration of the L-J algorithm in this case moves the parameter values away from the true values. The computational time is still less than the L-J method. We stress that although the L-J algorithm reaches the optimum in a fewer number of iterations, it requires much more CPU time than the least-squares algorithm, as can be seen from Table 1.

Table 1. Optimum Controller Tuning Parameters for a Process with the Transfer Function

$$G(s) = \frac{3.5e^{-0.5s}}{(5s+1)(2.5s+1)}$$

Method*	IAE	K_C	τ_I	τ_D	CPU Time (s)	No. Iterations
(a) Initial guesses: 1.0, 1.0, 1.0						
LSQ	6.2367	1.6826	7.3546	1.7460	2.454	18
L-J	6.2749	1.7161	7.4923	1.7939	13.146	10
Combined	6.2441	1.8111	7.324	1.7692	3.355	9
(b) Initial guesses: 5.0, 5.0, 5.0						
LSQ	6.2511	1.7992	7.3558	1.8078	3.577	29
L-J	6.2778	1.7038	7.2925	1.6764	11.508	10
Combined	6.2365	1.6846	7.3468	1.7463	2.721	10
(c) Initial guesses: 10.0, 10.0, 10.0						
LSQ	6.2438	1.7723	7.3115	1.7792	5.532	42
L-J	6.2932	1.6921	7.1922	1.9087	13.055	10
Combined	6.2434	1.7634	7.2942	1.7832	2.133	5
(d) Initial guesses: 0.1, 20.0, 0.1						
LSQ	6.2388	1.6861	7.3668	1.7445	2.340	14
L-J	6.2784	1.7958	7.4270	1.8409	13.349	10
Combined	6.2431	1.7814	7.3211	1.7795	4.574	22

*LSQ, least-squares method; L-J, Luus-Jaakola method

Conclusions

Optimal tuning parameters for a PID controller have been calculated using a least-squares parameter estimation method. This method is faster than other controller tuning algorithms because it exploits the specific structure of the objective function to be minimized. A variety of objective functions such as IAE, ITAE, and ISE can be minimized by this method. In combination with a process identification program, our method is suit-

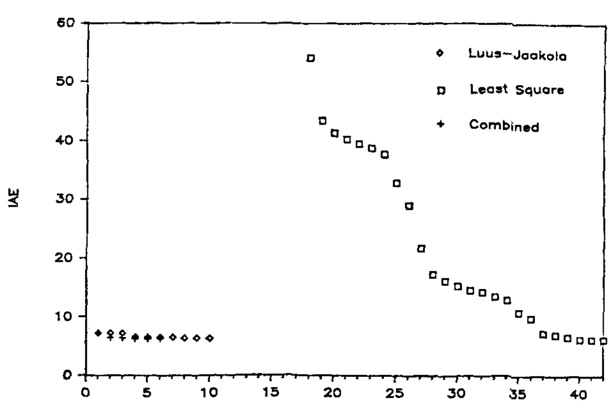

Figure 1. Progress toward convergence.

able for the on-line tuning of controllers, in view of the small computation times. It is an attractive option for controlling simple processes that do not warrant the use of more sophisticated algorithms such as internal model control, IMC (García and Morari, 1982).

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Notation

- e = error (set point − measured variable)
- g = difference between experimentally measured and predicted values; for a step change in set point, $g = 1 - e$
- G = column vector of M values of g
- I = $N \times N$ identity matrix
- J = Jacobian matrix
- K_C = proportional gain
- M = number of experimental points; for our problem, M = number of sampling instants for which closed-loop response is calculated
- N = number of adjustable parameters
- sgn = signum function
- x = vector of adjustable parameters
- y = sum of squares of the errors, $G^T G$

Greek letters

- λ = quantity, Eq. 8
- τ_D = derivative time
- τ_I = integral time

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